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EQUILIBRIUMS AND STABILITY OF AN SVIR EPIDEMIC MODEL

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ABSTRACT

An epidemic model is a simplified means of describing the transmission of communicable disease through individuals. Compartmental model is one of the easiest way to analyzed communicable diseases. In this paper a nonlinear mathematical deterministic compartmental SVIR model for the dynamics of infectious disease including the role of a preventive vaccine is proposed and analyzed. The model has various kinds of parameter such as natural birth rate, natural death rate and dieses related death rate. Also incoming immigrants are considered in this model. A model for the transmission dynamics of an infectious disease has been presented and analyzed the stability of equilibrium points of this model.

KEYWORDS: Endemic Equilibrium Basic Reproduction Number, Diseases Free Equilibrium, Infectious Diseases, Stability

INTRODUCTION

Mathematical modeling is one of the most important materials to analyze the characteristic of an infectious disease. One of the early triumphs of mathematical epidemiology was the formulation of a simple model by Kermack and McKendrick in 1927 [1]. The Kermack-McKendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population [2]. Various kinds of deterministic models for the spread of infectious disease have been analyzed by mathematical modeling to control the epidemic. Epidemiological models have two kinds of equilibrium points. One of them is disease free equilibrium (DFE) at which the population remains in the absence of disease and other is endemic equilibrium [3]. There are two major types of control strategies available to curtail the spread of infectious diseases: pharmaceutical interventions (drugs, vaccines etc) and non-pharmaceutical interventions (social distancing, quarantine). Vaccination is important for the elimination of infectious disease in pharmaceutical interventions. Arino et al introduced vaccination of susceptible individuals into an SIRS model and also considered vaccinating a fraction of newborns [4]. Buonomo et al studied the traditional SIR model with 100% efficacious vaccine [5]. The epidemic models with vaccination have been investigated recently by some authors [6−12]. Effective vaccines have been used successfully to control smallpox, polio and measles. In this paper an SIR type disease has been considered when a vaccination program is in effect.

MODEL FORMULATION

Let us now consider an SIR type disease when a vaccination program is in effect and there is a constant flow of incoming immigrants. We define $S(t)$, $V(t)$, $I(t)$, $R(t)$ and $N(t)$ be the number of susceptible, vaccinated, infective, recovered and total population respectively at time t. We model new infections using the simple mass-action law, so that in general there are αSI new infections in unit time when α is the rate of contact that is sufficient to transmit the disease. We also assume a constant recovery rate γ. The vaccine has the effect of reducing the susceptibility to infection by a

factor σ, so that $\sigma = 0$ means that the vaccine is completely effective in preventing infection, while σ = 1 means that the vaccine is utterly ineffective. The rate at which the susceptible population is vaccinated is ϕ. We assume that there can be disease related death and define α to be the rate of disease-related death, while μ is the rate of natural death that is not related to the disease. The population is replenished in two ways, birth and immigration. We assume that all newborns enter the susceptible class at the constant rate of Λ and there is a constant incoming flow of immigrants, A, where some portion p of immigrants are infective. We can now formulate this model, dividing the population into four classes – susceptible (S), vaccinated (V), infective (I) and recovered (R).

In summary, the assumptions we have in this model is as follows:

- $S(t)$, $I(t)$, $V(t)$, $R(t)$ and $N(t)$ are the numbers of susceptible, infective, vaccinated, recovered and total population at time *t* respectively.
- There is a constant flow A of new members into the population per unit time, where fraction p of immigrants is infective $(0 \le p \le 1)$.
- The vaccine has effect of reducing infection by a factor of σ, so that $\sigma = 0$ means that the vaccine is completely effective in preventing infection, while $\sigma = 1$ means that the vaccine is utterly ineffective.
- ϕ is the rate at which the susceptible population is vaccinated.
- The disease can be fatal to some infective and we define β to be the rate of disease related death.

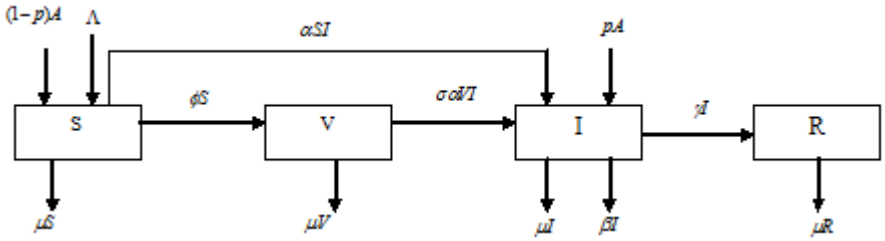

Figure 1: Diagram of SVIR Model

- There is a constant per capita natural death rate $\mu > 0$.
- Fraction $\gamma \ge 0$ of infective recovers in unit time.
- αN is the infectious contact rate per person in unit time.
- Λ is the constant natural birth rate, with all newborns coming into the susceptible class.

The differential equations of this model are given by

$$\left.\begin{aligned} S' &= (1-p)A + \Lambda - \alpha SI - (\mu + \phi)S \\ V' &= \phi S - \sigma\alpha VI - \mu V \\ I' &= pA + \alpha SI + \sigma\alpha VI - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \end{aligned}\right\} \quad (1)$$

Note that the total population is the sum of the classes: susceptible, infective, vaccinated and recovered, i.e,

$$N(t) = S(t) + V(t) + I(t) + R(t) \tag{2}$$

So, $N'(t) = S'(t) + V'(t) + I'(t) + R'(t)$

Using (1) and (2) we get

$\Rightarrow \quad N' = A + \Lambda - \mu N - \beta I$

We can get an alternate but yet equivalent model by replacing *S* with *N-V-I-R*. Now the model becomes:

$$\begin{aligned} V' &= \phi(N - I - R) - \sigma\alpha VI - (\mu + \phi)V \\ I' &= pA + \alpha(N - (1-\sigma)V - I - R)I - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \\ N' &= A + \Lambda - \mu N - \beta I \end{aligned} \tag{3}$$

The Jacobean matrix of the above system is

$$J = \begin{bmatrix} \frac{\partial}{\partial V}(V') & \frac{\partial}{\partial I}(V') & \frac{\partial}{\partial R}(V') & \frac{\partial}{\partial N}(V') \\ \frac{\partial}{\partial V}(I') & \frac{\partial}{\partial I}(I') & \frac{\partial}{\partial R}(I') & \frac{\partial}{\partial N}(I') \\ \frac{\partial}{\partial V}(R') & \frac{\partial}{\partial I}(R') & \frac{\partial}{\partial R}(R') & \frac{\partial}{\partial N}(R') \\ \frac{\partial}{\partial V}(N') & \frac{\partial}{\partial I}(N') & \frac{\partial}{\partial R}(N') & \frac{\partial}{\partial N}(N') \end{bmatrix}$$

i.e., $J = \begin{bmatrix} -\sigma\alpha I - \mu - \phi & -\phi - \sigma\alpha V & -\phi & \phi \\ (\sigma - 1)\alpha I & -2\alpha I + \alpha(N - (1-\sigma)V - R) - (\mu + \gamma + \beta) & -\alpha I & \alpha I \\ 0 & \gamma & -\mu & 0 \\ 0 & -\beta & 0 & -\mu \end{bmatrix}$

EQUILIBRIUM CONDITIONS

We can write the equilibrium conditions by letting the right hand side equations of (3) to be zero. The equilibrium conditions are

$$\phi(N - I - R) - \sigma\alpha VI - (\mu + \phi)V = 0 \tag{4}$$

$$pA + \alpha(N - (1-\sigma)V - I - R)I - (\mu + \gamma + \beta)I = 0 \tag{5}$$

$$\gamma I - \mu R = 0 \tag{6}$$

$$A + \Lambda - \mu N - \beta I = 0 \tag{7}$$

From (7) we get

$$N = \frac{A + \Lambda - \beta I}{\mu}$$

Again from (6) we get

$$R = \frac{\gamma I}{\mu}$$

Solving (4) for V and substituting the values of *N* and *R* we get

$$V = \frac{\phi\left(A + \Lambda - (\beta + \mu + \gamma)I\right)}{\mu\left(\alpha\sigma I + \mu + \phi\right)}$$

Eliminating N,V and R by substituting these values in (5) and simplifying we get a cubic equation for the equilibrium values of *I* of the form

$$EI^3 + BI^2 + CI + D = 0 \tag{8}$$

where $E = -\alpha^2\sigma(\gamma + \mu + \beta)$

$$B = -\alpha(\phi\sigma\gamma + \mu\sigma\gamma + \mu\gamma - \alpha\sigma A + \mu\phi\sigma + \mu^2\sigma + \beta\mu\sigma - \alpha\Lambda\sigma + \mu^2 + \beta\mu)$$

$$C = -(\phi + \mu)\mu\gamma + \left((\phi + \mu p)\sigma + \mu\right)\alpha A + \alpha\phi\sigma\Lambda - (\mu + \beta)\mu\phi - \mu^3 - \beta\mu^2 + \alpha\mu\Lambda$$

$$D = \alpha\phi\sigma A - (\phi + \mu)\mu\gamma + \alpha\mu\sigma pA + \alpha\mu A + \alpha\sigma\phi\Lambda - \mu^2\phi - \beta\mu\phi - (\mu + \beta)\mu^2 + \alpha\mu\Lambda$$

VACCINE REPRODUCTIVE NUMBER

Let us consider the case when there are no infective immigrants, i.e. $p = 0$. So From (3) we get

$$\begin{aligned} V' &= \phi(N - I - R) - \sigma\alpha VI - (\mu + \phi)V \\ I' &= \alpha(N - (1-\sigma)V - I - R)I - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \\ N' &= A + \Lambda - \mu N - \beta I \end{aligned} \tag{9}$$

We can write the equilibrium conditions by letting the right-hand side of (9) be zero. The equilibrium conditions are.

$$\phi(N - I - R) - \sigma\alpha VI - (\mu + \phi)V = 0 \tag{10}$$

$$\alpha(N - (1-\sigma)V - I - R)I - (\mu + \gamma + \beta)I = 0 \tag{11}$$

$$\gamma I - \mu R = 0 \tag{12}$$

$$A + \Lambda - \mu N - \beta I = 0 \tag{13}$$

From (13) we get

$$N = \frac{A + \Lambda - \beta I}{\mu} \tag{14}$$

Again from (12) we get

$$R = \frac{\gamma I}{\mu} \tag{15}$$

From (11) we can easily see that there is disease-free equilibrium, $I=0$.

At disease-free equilibrium (DFE) we can evaluate the other equilibrium values of *R* and *N* using (14) and (15)

$$R = 0\ ,\ N = \frac{A + \Lambda}{\mu}$$

Solving (10) for V and substituting The values of N and R we get

$$V = \frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}$$

So, the DFE $P_0(V, I, R, N) = \left(\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}, 0, 0, \frac{A+\Lambda}{\mu} \right)$

Now the Jacobian matrix at P_0 is

$$J_0 = \begin{bmatrix} -\mu-\phi & -\phi-\sigma\alpha\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)} & -\phi & \phi \\ 0 & \alpha\left(\frac{A+\Lambda}{\mu}-(1-\sigma)\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}\right)-(\mu+\gamma+\beta) & 0 & 0 \\ 0 & \gamma & -\mu & 0 \\ 0 & -\beta & 0 & -\mu \end{bmatrix}$$

The eigenvalues of the above matrix are

$$\lambda_{1,2} = -\mu$$

$$\lambda_3 = -(\mu+\phi)$$

$$\lambda_4 = \alpha\left(\frac{A+\Lambda}{\mu}-(1-\sigma)\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}\right)-(\mu+\gamma+\beta)$$

Since for positive parameter $\lambda_{1,2}$ and λ_3 are negative, the only condition for stability of DFE is $\lambda_4 < 0$.

i.e., $$\alpha\left(\frac{A+\Lambda}{\mu}-(1-\sigma)\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}\right)-(\mu+\gamma+\beta<0$$

$$\Rightarrow \quad \alpha\left(\frac{A+\Lambda}{\mu}-(1-\sigma)\frac{\phi(A+\Lambda)}{\mu(\mu+\phi)}\right)<(\mu+\gamma+\beta$$

$$\Rightarrow \quad \frac{(\alpha\phi\sigma+\alpha\mu)(A+\Lambda)}{\mu(\mu+\phi)}<(\mu+\gamma+\beta$$

$$\Rightarrow \quad \frac{(\alpha\phi\sigma+\alpha\mu)(A+\Lambda)}{\mu(\mu+\phi)(\mu+\gamma+\beta)}<1$$

Thus we can define a vaccine reproductive number $R(\phi) = \frac{(\alpha\phi\sigma+\alpha\mu)(A+\Lambda)}{\mu(\mu+\phi)(\mu+\gamma+\beta)}$ and the DFE is locally asymptotically stable iff $R(\phi)<1$. In the absence of vaccine (here $\phi = 0$), we define the basic reproductive number as $R_0 = \frac{\alpha(A+\Lambda)}{\mu(\mu+\gamma+\beta)}$.

WHEN THERE ARE NO INFECTIVE IMMIGRANTS

For the Case $\sigma = 1$ and $p = 0$:

Let us first consider the two extreme cases in order to investigate endemic equilibrium for the model (3). The first

case is when the vaccine is useless and there is no infective immigrant, i.e. $\sigma = 1$ and $p = 0$. Our system becomes

$$\begin{aligned} S' &= A + \Lambda - \alpha SI - (\mu + \phi)S \\ V' &= \phi S - \alpha VI - \mu V \\ I' &= \alpha SI + \alpha VI - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \end{aligned}$$

with $N(t) = S(t) + V(t) + I(t) + R(t)$. Note that $R(\phi)$ reduces to R_0 when $\sigma = 1$. By substitution and using the equilibrium conditions, one can find there is an endemic equilibrium

$I^* = \dfrac{\alpha(A+\Lambda) - \mu(\beta+\gamma+\mu)}{\alpha(\beta+\gamma+\mu)}$ which exist only when $R_0 = \dfrac{\alpha(A+\Lambda)}{\mu(\mu+\gamma+\beta)} > 1$

For the Case $\sigma = 0$ and $p = 0$:

If we suppose that the vaccine is completely effective and that there are no infective immigrants, i.e. $\sigma = 0$ and $p = 0$, then the model becomes

$$\begin{aligned} S' &= A + \Lambda - \alpha SI - (\mu + \phi)S \\ V' &= \phi S - \mu V \\ I' &= \alpha SI - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \end{aligned}$$

with $N(t) = S(t) + V(t) + I(t) + R(t)$. Note After complicated calculation we find that there is one endemic equilibrium

$I^* = \dfrac{\alpha(A+\Lambda)\mu - \mu(\phi+\mu)(\beta+\gamma+\mu)}{\alpha(\beta+\gamma+\mu)\mu}$ which exist only when $R_0 = \dfrac{\alpha(A+\Lambda)}{\mu(\mu+\gamma+\beta)} > 1$

For the Case $0 < \sigma < 1$ and $p = 0$:

Now we wish to consider the more general case when the vaccine is partially ineffective and when there are no infective immigrant, i.e. $0 < \sigma < 1$ and $p = 0$. Then the model becomes

$$\left.\begin{aligned} S' &= A + \Lambda - \alpha SI - (\mu + \phi)S \\ V' &= \phi S - \sigma\alpha VI - \mu V \\ I' &= \alpha SI + \sigma\alpha VI - (\mu + \gamma + \beta)I \\ R' &= \gamma I - \mu R \end{aligned}\right\} \qquad (16)$$

with $N(t) = S(t) + V(t) + I(t) + R(t)$. From equilibrium condition (8), we know that there is a disease-free equilibrium regardless of different parameter values. After factoring out this disease-free equilibrium, *I*, we get an equilibrium condition as a quadratic polynomial of I as

$$EI^2 + BI + C = 0 \qquad (17)$$

where $E == \alpha^2\sigma(\gamma + \mu + \beta)$

$$B = \alpha(\phi\sigma\gamma + \mu\sigma\gamma + \mu\gamma - \alpha\sigma A + \mu\phi\sigma + \mu^2\sigma + \beta\mu\sigma - \alpha\Lambda\sigma + \mu^2 + \beta\mu)$$

$$C = (\phi + \mu)\mu\gamma - \big((\phi + \mu p)\sigma + \mu\big)\alpha A - \alpha\phi\sigma\Lambda + (\mu + \beta)\mu\phi + \mu^3 + \beta\mu^2 - \alpha\mu\Lambda$$

Now an endemic equilibrium corresponds to a positive real solution of equation (17). Note that E > 0 and that C< 0 precisely when $R(\phi)$ > 1. Note also that $B^2 - 4EC > 0$ when C < 0. One can easily deduce that there is precisely one endemic equilibrium when $R(\phi) > 1$, since there are two real roots and the product of those two roots is negative. On the other hand we can see that C > 0 if $R(\phi) < 1$. Note that there are exactly two changes in the sign of coefficients of equation (17) if coefficient B < 0 and none when B > 0. By Descartes' rule of signs one can conclude that the maximum number of endemic equilibrium is two when $R(\phi) < 1$ and B < 0, and that there is no endemic equilibrium when $R(\phi) < 1$ and B > 0. However it is shown that it is always the case that the system does not have any endemic equilibrium when $R(\phi) < 1$.

Proposition 1: For model (16) with $R(\phi) = \dfrac{(\alpha\phi\sigma + \alpha\mu)(A+\Lambda)}{\mu(\mu+\phi)(\mu+\gamma+\beta)}$ there is no endemic equilibrium when $R(\phi) < 1$.

Proof : We first assume that $R(\phi) < 1$ and $B < 0$:

$$R(\phi) < 1 \Leftrightarrow (\alpha\phi\sigma + \alpha\mu)(A+\Lambda) < \mu(\mu+\phi)(\mu+\gamma+\beta)$$

$$B < 0 \Leftrightarrow \alpha\sigma\mu(\beta+\gamma+\mu) + \alpha(\beta+\gamma+\mu)(\mu+\sigma\phi) < \alpha^2\sigma(A+\Lambda)$$

Combining above two conditions one can get the following relation.

$$\sigma\mu(\beta+\gamma+\mu) + (\beta+\gamma+\mu)(\mu+\sigma\phi) < \frac{\sigma\mu(\beta+\gamma+\mu)(\mu+\phi)}{\mu+\sigma\phi}$$

After expansion and some calculations we get

$$\sigma\phi(\mu+\sigma\phi)\mu^2 + \sigma^2\mu\phi < 0$$

This is a contradiction for all non negative parameters. Therefore $R(\phi) < 1$ and $B < 0$ is impossible. So when $R(\phi) < 1, B > 0$. Note that $R(\phi) < 1$ corresponds to $C > 0$. Also clearly $E > 0$. So by Descartes rule of signs there is no endemic equilibrium for $R(\phi) < 1$.

WHEN THERE ARE NO DISEASE FATALITIES BUT INFECTIVE IMMIGRANTS

In this section we assume that there is a constant flow of infective immigrants into host population and no disease fatality, i.e., $\beta = 0$ and $p \neq 0$. Based on this assumption, the model equations becomes

$$\begin{aligned} S' &= (1-p)A + \Lambda - \alpha SI - (\mu+\phi)S \\ V' &= \phi S - \sigma\alpha VI - \mu V \\ I' &= pA + \alpha SI + \sigma\alpha VI - (\mu+\gamma)I \\ R' &= \gamma I - \mu R \end{aligned}$$

with $N(t) = S(t) + V(t) + I(t) + R(t)$.

So $N'(t) = S'(t) + V'(t) + I'(t) + R'(t) = A + \Lambda - \mu N$

Now by theory of autonomous system $\lim_{t \to \infty} N'(t) = 0$, So $\lim_{t \to \infty} N(t) = \dfrac{A+\Lambda}{\mu} = K$ (say)

So, replacing *S* by *K-I-V-R* we have reduced system of equations

$$\begin{aligned} V' &= \phi(K - I - V - R) - \sigma\alpha VI - \mu V \\ I' &= pA + \alpha(K - I - V - R)I + \sigma\alpha VI - (\mu + \gamma)I \\ R' &= \gamma I - \mu R \end{aligned} \tag{18}$$

Equilibrium Conditions

Endemic equilibrium conditions are

$$\begin{aligned} \phi(K - I - V - R) &= \sigma\alpha VI + \mu V \\ \frac{pA}{I} &= (\mu + \gamma) - \sigma\alpha V - \alpha(K - V - I - R) \\ \gamma I &= \mu R \end{aligned}$$

One can reduce these endemic equilibrium conditions into one cubic equation of I by substituting for K,V and R:

$$f(I) = EI^3 + BI^2 + CI + D = 0$$

where $E = \alpha^2\sigma(\mu + \gamma)$

$$B = \alpha\left[-\alpha\sigma(A + \Lambda) + (\mu + \gamma)(\mu + \sigma\phi + \sigma\mu)\right]$$

$$C = -pA\mu\sigma\alpha - \alpha(A + \Lambda)(\mu + \alpha\phi) + \mu(\mu + \gamma)(\mu + \phi)$$

$$D = -pA\mu^2(\mu + \phi)$$

Since $D < 0$ for positive parameter, then $f(o) < 0$. Also $\underset{I \to \infty}{Lim} f(I) = \infty$, So there exist one or three positive roots I^*. Now let us consider $f'(I) = 3EI^2 + 2BI + C$. By Rolle's theorem, if $f(I) = 0$ has three distinct positive roots then $f'(I) = 0$ must have two distinct positive roots. $B < 0$ and $C > 0$ is necessary condition to three positive endemic equilibriums.

Stability Analysis

The Jacobean matrix of the system (18) is

$$J = \begin{bmatrix} -\sigma\alpha I - \phi - \mu & -\sigma\alpha V - \phi & -\phi \\ \sigma\alpha I - \alpha I & \sigma\alpha V - \gamma + \alpha(K - V - I - R) - \alpha I - \mu & -\alpha I \\ 0 & \gamma & -\mu \end{bmatrix}$$

Using equilibrium condition $\frac{pA}{I} = (\mu + \gamma) - \sigma\alpha V - \alpha(K - V - I - R)$ we get

$$J = \begin{bmatrix} -\sigma\alpha I - \phi - \mu & -\sigma\alpha V - \phi & -\phi \\ \sigma\alpha I - \alpha I & -\frac{pA}{I} - \alpha I & -\alpha I \\ 0 & \gamma & -\mu \end{bmatrix}$$

The characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Where

$$a_1 = \sigma\alpha I + \alpha I + \frac{pA}{I} + \phi + 2\mu$$

$$a_2 = \alpha\gamma I + \alpha^2\sigma(\sigma-1)IV + \alpha^2\sigma I^2 + \big((\alpha\phi + \alpha\mu)\sigma + 2\alpha\mu\big)I + \alpha\sigma pA + \mu\phi + \mu^2 + \frac{(p\phi + 2\mu p)A}{I}$$

$$a_3 = \big(\alpha^2\sigma I^2 + (\alpha\sigma\phi + \alpha\mu)I\big)\gamma + \alpha^2\sigma\mu(\sigma-1)IV + \alpha^2\sigma\mu I^2 + (\alpha\mu\phi\sigma + \alpha\mu^2)I + \alpha\mu\sigma pA + \frac{(\mu\phi p + \mu^2 p)A}{I}$$

By the Routh-Hurwitz Criterion [13], the endemic equilibrium is globally stable if and only if

$a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$

CONCLUSIONS

We have formulated an epidemic model with vaccination and investigated their dynamical behaviors. By means of the Jacobean matrix, we obtained their vaccine reproductive number and basic reproduction number, which play a crucial role. It has been observed that the DFE is locally asymptotically stable if and only if the vaccine reproductive number is less than one. When the vaccine is useless or completely effective and there is no infective immigrant, the endemic equilibrium is exist if and only if the basic reproduction number is greater than one. When there are no disease fatalities but with infective immigrants we give the condition for endemic equilibrium and their stability condition.

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